

# MODULAR VARIETIES OF $\mathcal{D}$ -ELLIPTIC SHEAVES AND THE WEIL-DELIGNE BOUND

MIHRAN PAPIKIAN

**ABSTRACT.** We compare the asymptotic growth of the number of rational points on modular varieties of  $\mathcal{D}$ -elliptic sheaves over finite fields to the growth of their Betti numbers as the degree of the level tends to infinity. This is a generalization to higher dimensions of a well-known result for modular curves. As a consequence of the main result, we also produce a new asymptotically optimal sequence of curves.

## 1. INTRODUCTION

**1.1. Motivation.** Let  $q$  be a power of a prime  $p$  and let  $\mathbb{F}_q$  denote the finite field with  $q$  elements. Let  $X$  be a smooth, projective, geometrically irreducible,  $d$ -dimensional variety over  $\mathbb{F}_q$ . Fix an algebraic closure  $\overline{\mathbb{F}}_q$  of  $\mathbb{F}_q$ . Also, fix a prime number  $\ell \neq p$  and an algebraic closure  $\overline{\mathbb{Q}}_\ell$  of the field  $\mathbb{Q}_\ell$  of  $\ell$ -adic numbers. Grothendieck's theory of étale cohomology produces the  $\ell$ -adic cohomology groups

$$H^i(X) := H^i(X \otimes_{\mathbb{F}_q} \overline{\mathbb{F}}_q, \overline{\mathbb{Q}}_\ell), \quad i \geq 0.$$

These groups are finite dimensional  $\overline{\mathbb{Q}}_\ell$ -vector spaces endowed with an action of the Galois group  $\text{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ . The  $\ell$ -adic Betti numbers of  $X$  are the dimensions

$$h^i(X) := \dim_{\overline{\mathbb{Q}}_\ell} H^i(X).$$

It is known that  $h^i(X) = 0$  for  $i > 2d$ ,  $h^0(X) = 1$ , and  $h^{2d-i}(X) = h^i(X)$ .

Let  $\text{Frob}_q$  be the inverse of the standard topological generator  $x \mapsto x^q$  of  $\text{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ , i.e., the so-called *geometric Frobenius element*. Assume  $H^i(X) \neq 0$ . Denote the eigenvalues of  $\text{Frob}_q$  acting on  $H^i(X)$  by  $\alpha_{i,1}, \alpha_{i,2}, \dots, \alpha_{i,s}$  (here  $s = h^i(X)$ ). Deligne proved that  $\{\alpha_{i,j}\}$  are algebraic numbers. Moreover, for any isomorphism  $\iota : \overline{\mathbb{Q}}_\ell \rightarrow \mathbb{C}$  the absolute value  $|\iota(\alpha_{i,j})|$  is independent of  $\iota$  and is equal to  $q^{i/2}$  (Riemann hypothesis for  $X$ ); see [5].

For an integer  $n \geq 1$  denote by  $\mathbb{F}_{q^n}$  the degree  $n$  extension of  $\mathbb{F}_q$ , and let  $X(\mathbb{F}_{q^n})$  be the set of  $\mathbb{F}_{q^n}$ -rational points on  $X$ . By the Grothendieck-Lefschetz trace formula

$$\#X(\mathbb{F}_{q^n}) = \sum_{i=0}^{2d} (-1)^i \text{Tr}(\text{Frob}_q^n | H^i(X)) = \sum_{i=0}^{2d} (-1)^i \sum_{j=1}^{h^i(X)} \alpha_{i,j}^n.$$

If one combines this with Deligne's result, then there results the *Weil-Deligne bound*

$$(1.1) \quad \#X(\mathbb{F}_{q^n}) \leq \sum_{i=0}^{2d} q^{in/2} h^i(X) =: \text{WD}_n(X).$$

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It is natural to ask how “optimal” is the bound (1.1). More precisely, suppose we fix some natural numbers  $b_1, b_2, \dots, b_d$ . *How close can  $\#X(\mathbb{F}_{q^n})$  get to  $\text{WD}_n(X)$  for a variety  $X$  with  $h^i(X) = b_i$ ,  $1 \leq i \leq d$ ?* Although this question has received a considerable amount of attention, it still remains largely open, cf. [27], [30].

Let  $h(X) := \sum_{i=0}^{2d} h^i(X)$ . In this paper we will be mostly concerned with the asymptotic optimality of (1.1): *How close can the ratio  $\#X(\mathbb{F}_{q^n})/h(X)$  get to  $\text{WD}_n(X)/h(X)$  as  $h(X) \rightarrow \infty$ , assuming  $d, q$  and  $n$  are fixed?* Tsfasman raised questions of this nature in [30]. Besides its intrinsic mathematical interest, the motivation for this problem partly comes from coding theory via Goppa’s [11] algebro-geometric construction of error-correcting codes over  $\mathbb{F}_{q^n}$ .

For curves (i.e., when  $d = 1$ ) we clearly have  $\text{WD}_n(X)/h(X) \rightarrow q^{n/2}$  as  $h(X) \rightarrow \infty$ . On the other hand, surprisingly enough, it turns out that

$$(1.2) \quad \limsup_X \left( \frac{\#X(\mathbb{F}_{q^n})}{h(X)} \right) \leq \frac{q^{n/2} - 1}{2}.$$

This is a well-known result of Drinfeld and Vladut [32]. In particular, curves of genus sufficiently larger than  $q^n$  never have as many rational points as the Weil-Deligne bound allows.

A sequence of curves  $\{X_i\}$  over  $\mathbb{F}_{q^n}$  is called *asymptotically optimal* if  $h(X_i) \rightarrow \infty$  and  $\#X_i(\mathbb{F}_{q^n})/h(X_i) \rightarrow (q^{n/2} - 1)/2$ . It is not known whether asymptotically optimal sequences exist when  $q^n$  is not a square (even for a single  $q^n$ ), in other words, it is an open problem whether in general (1.2) is the best possible upper-bound on  $\limsup(\#X(\mathbb{F}_{q^n})/h(X))$ . On the other hand, when  $q^n$  is a square, then asymptotically optimal sequences always exist. Here Shimura curves (and their function field analogues - Drinfeld modular curves) play a key role: modular curves with appropriate level structures over quadratic extensions of finite fields attain the bound (1.2) as the size of the level tends to infinity. This is due to Drinfeld, Ihara, Manin, Tsfasman, Vladut and Zink; see [14], [19], [31], [32]. In fact, every known asymptotically optimal sequence of curves over  $\mathbb{F}_{q^2}$  has the property that every  $X_i$  is a classical, Shimura or Drinfeld modular curve for  $i$  sufficiently large, cf. [8], [9].

For the higher dimensional varieties there are only a few partial results. Lachaud and Tsfasman, using explicit formulae, proved a certain analogue of the Drinfeld-Vladut bound (1.2); see [15], [30]. As far as I am aware, there were no known examples of sequences of  $d$ -dimensional varieties  $\{X_i\}$  such that  $h(X_i) \rightarrow \infty$  and for which  $\#X_i(\mathbb{F}_q)/h(X_i)$  converges to a number close to the asymptotic Weil-Deligne bound (assuming  $q$  is fixed), besides the following obvious construction. Take each  $X_i$  to be an appropriate product of lower dimensional varieties. The number of rational points and the Betti numbers are easy to compute inductively. Indeed, if  $X = Y \times Z$  then  $\#X(\mathbb{F}_q) = (\#Y(\mathbb{F}_q)) \cdot (\#Z(\mathbb{F}_q))$  and  $h(X) = h(Y) \cdot h(Z)$ . (The first identity is clear and the second follows from Künneth formula.) Using this technique, one can produce all sorts of interesting limits (but they all are smaller than the asymptotic Weil-Deligne bounds) cf. [30, §5].

The main result of this paper is a generalization of Tsfasman-Vladut-Zink result for Shimura curves to the case of certain higher dimensional modular varieties. The question of extending the results in [31] to other modular varieties already appears in that paper (Question C on p.22).

*Remark 1.1.* One might ask whether the Weil-Deligne bound is ever asymptotically optimal. More precisely, suppose  $d$  and  $q$  are fixed. Does there exist a sequence  $\{X_i\}$  of  $d$ -dimensional varieties over  $\mathbb{F}_q$  such that  $h(X_i) \rightarrow \infty$  and  $\#X_i(\mathbb{F}_q)/\text{WD}_1(X_i) \rightarrow 1$ ? I expect that the answer is always negative. When  $d = 1$  this of course follows from (1.2).

**1.2. Main result.** Let  $C := \mathbb{P}_{\mathbb{F}_q}^1$  be the projective line over  $\mathbb{F}_q$ . Denote by  $F = \mathbb{F}_q(T)$  the field of rational functions on  $C$ . Fix some  $d \geq 2$  and let  $D$  be a  $d^2$ -dimensional central division algebra over  $F$ , which is split at  $\infty = 1/T$ , i.e.,  $D \otimes_F F_\infty$  is isomorphic to the algebra  $\mathbb{M}_d(F_\infty)$  of  $d \times d$  matrices with entries in  $F_\infty$ . Fix a locally-free sheaf  $\mathcal{D}$  of  $\mathcal{O}_C$ -algebras on  $C$  whose generic fibre is  $D$  and such that for every place  $x \in |C|$ ,  $\mathcal{D}_x := \mathcal{D} \otimes_{\mathcal{O}_C} \mathcal{O}_x$  is a maximal order of  $D_x = D \otimes_F F_x$ , where  $\mathcal{O}_x$  and  $F_x$  are the completions of  $\mathcal{O}_C$  and  $F$  at  $x$ , respectively. Denote by  $R \subset |C|$  the set of places where  $D$  ramifies; hence for all  $x \notin R$  the couple  $(D_x, \mathcal{D}_x)$  is isomorphic to  $(\mathbb{M}_d(F_x), \mathbb{M}_d(\mathcal{O}_x))$ . *In this paper we make a blanket assumption that  $D_x$  is a division algebra for every  $x \in R$ .* Let  $o$  be a fixed closed point on  $C - R - \{\infty\}$ . Denote the residue field at  $o$  by  $\mathbb{F}_o$ , its degree  $m$  extension by  $\mathbb{F}_o^{(m)}$ , and  $q_o := q^{[\mathbb{F}_o : \mathbb{F}_q]} = \#\mathbb{F}_o$ .

Call a prime ideal  $\mathfrak{p}$  of the polynomial ring  $A := \mathbb{F}_q[T]$  *admissible* if  $x \mapsto x^d$  is an automorphism of  $(A/\mathfrak{p})^\times / \mathbb{F}_q^\times$ . There are infinitely many admissible primes; see [21, Lem. 4.6]. A closed subscheme  $I$  of  $C$  is called *admissible* if  $I \cap (R \cup \infty \cup o) = \emptyset$  and  $I = \text{Spec}(A/\mathfrak{p})$  for an admissible prime  $\mathfrak{p} \triangleleft A$ .

Assume  $I = \text{Spec}(A/\mathfrak{p})$  is admissible and let  $M_{I,o}^{\mathcal{D}}$  be the modular variety of isomorphism classes of  $\mathcal{D}$ -elliptic sheaves over  $\overline{\mathbb{F}}_o$  with pole at  $\infty$  equipped with level- $I$  structures; see §3 for definitions. There is a natural action of  $(A/\mathfrak{p})^\times$  on  $M_{I,o}^{\mathcal{D}}$  via its (diagonal) action on level structures. We denote the quotient variety by  $\bar{M}_{I,o}^{\mathcal{D}}$ . The main result of this paper is the following:

**Theorem 1.2.**  *$\bar{M}_{I,o}^{\mathcal{D}}$  is a smooth, projective, geometrically irreducible,  $(d-1)$ -dimensional variety defined over  $\mathbb{F}_o$ . Moreover*

$$(1.3) \quad \lim_{\deg(I) \rightarrow \infty} \frac{\#\bar{M}_{I,o}^{\mathcal{D}}(\mathbb{F}_o^{(d)})}{h(\bar{M}_{I,o}^{\mathcal{D}})} = \frac{1}{d} \prod_{i=1}^{d-1} (q_o^i - 1).$$

A similar result for Drinfeld modular varieties is proven in [21].

It will be clear from the proof of Theorem 1.2 that  $h(\bar{M}_{I,o}^{\mathcal{D}}) \rightarrow \infty$  as  $\deg(I) \rightarrow \infty$ . If we specialize the theorem to  $d = 2$  and  $\deg(o) = 1$ , then we get a sequence of curves over  $\mathbb{F}_{q^2}$ , indexed by the primes in  $A$  of odd degree, which is asymptotically optimal. This last fact is new and is of independent interest due to a rather limited number of examples of asymptotically optimal sequences of curves.

From one of the main results in [18] (see Theorem 6.1) or, alternatively, from the discussion in §6.2, it follows that

$$(1.4) \quad \lim_{\deg(I) \rightarrow \infty} \frac{\text{WD}_d(\bar{M}_{I,o}^{\mathcal{D}})}{h(\bar{M}_{I,o}^{\mathcal{D}})} = q_o^{d(d-1)/2}.$$

One can compare the limits (1.3) and (1.4) from two opposite viewpoints. On the one hand, since  $\frac{1}{d} \prod_{i=1}^{d-1} (q_o^i - 1) < q_o^{d(d-1)/2}$ , modular varieties  $\bar{M}_{I,o}^{\mathcal{D}}$  never have as many  $\mathbb{F}_o^{(d)}$ -rational points as the Weil-Deligne bound allows when  $\deg(I)$  is large. On the other hand, the degree of  $\frac{1}{d} \prod_{i=1}^{d-1} (q_o^i - 1)$  as a polynomial in  $q_o$  is the same as

the degree of  $q_o^{d(d-1)/2}$ , so  $\#\bar{M}_{I,o}^{\mathcal{P}}(\mathbb{F}_o^{(d)})$  asymptotically comes close to  $\text{WD}_d(\bar{M}_{I,o}^{\mathcal{P}})$ , especially when  $q_o$  is large compared to  $d$ , and one can say that the varieties  $\bar{M}_{I,o}^{\mathcal{P}}$  have many  $\mathbb{F}_o^{(d)}$ -rational points compared to their Betti numbers.

**1.3. Shimura varieties.** Modular varieties of  $\mathcal{D}$ -elliptic sheaves are the function field analogues of Shimura varieties over number fields. The anonymous referee pointed out that in a recent paper [24] François Sauvageot considered a problem about asymptotic properties of Shimura varieties which is similar to the problem we address in this paper. In this subsection we would like to compare Sauvageot's result to ours.

Let  $G$  be an algebraic connected reductive group over a global field of characteristic 0. Sauvageot introduces a notion of *strongly vanishing family*  $\mathcal{K}$  of compact subgroups of  $G$  over the finite adeles. Attached to  $G$  and  $K \in \mathcal{K}$  there is a Shimura variety  $X_K$ . Generalizing an approach of Serre [28] by means of Arthur's trace formula, Sauvageot gives a simple expression for  $\lim_{K \in \mathcal{K}} \text{Vol}(K) \chi(X_K)$  in terms of an invariant of  $G$  and  $\mathcal{K}$ ; here  $\chi(X_K)$  is the Euler-Poincaré characteristic of the  $L^2$ -cohomology of  $X_K$  and  $\text{Vol}(K)$  is the volume of  $K$  with respect to an appropriately normalized Haar measure. On the other hand, for a prime of good reduction  $o$  of  $X_K$ , assuming a conjecture of Milne, a formula of Kottwitz expresses  $\lim_{K \in \mathcal{K}} \text{Vol}(K) (\#X_{K,o}(\mathbb{F}_o^{(m)}))$ ,  $m \geq 1$ , in terms of an invariant of  $G$  and a sum of twisted orbital integrals. This again uses trace formula techniques. Overall, one obtains a (conjectural) formula for  $\lim_{K \in \mathcal{K}} \#X_{K,o}(\mathbb{F}_o^{(m)}) / \chi(X_K)$  which involves a sum of twisted orbital integrals. Then Sauvageot gives some criteria for detecting strongly vanishing families  $\mathcal{K}$  (e.g., for  $\text{GL}_2$  over  $\mathbb{Q}$  the classical congruence subgroups form such families as the level tends to infinity). Some relevant questions are not discussed in [24]: First, whether it is possible to compute  $\lim_{K \in \mathcal{K}} \#X_{K,o}(\mathbb{F}_o^{(m)}) / \chi(X_K)$  explicitly in some situations other than the case of curves, i.e., to give a simple expression for the sum of twisted orbital integrals which comes from Kottwitz's formula. Second, what is the relationship of  $\chi(X_K)$  to the invariants of  $X_{K,o}$ , such as the  $\ell$ -adic Euler-Poincaré characteristic of  $X_{K,o}$ , especially when  $X_K$  is not compact. Third, for which  $K$  the resulting varieties  $X_{K,o}$  are smooth and geometrically irreducible.

Now from the point of view of general Shimura varieties the situation which we consider in this paper is rather special since the only  $G$  which is allowed is the multiplicative group of a division algebra. Our notion of admissible level seems to be analogous to Sauvageot's notion of vanishing family. The main advantage of our result is that the asymptotic formula which we obtain is very explicit. Our strategy of the proof is also different. In computing the asymptotic number of rational points over finite fields we crucially rely on the modular interpretation of the points on  $M_{I,o}^{\mathcal{P}}$ . In computing the asymptotic Betti numbers we use special properties of the cohomology of varieties having non-archimedean uniformization, along with some classical results about discrete subgroups of  $p$ -adic groups. Note that estimating the sum of Betti numbers  $h(\bar{M}_{I,o}^{\mathcal{P}})$  or their alternating sum  $\chi(\bar{M}_{I,o}^{\mathcal{P}})$  are equivalent problems for  $\bar{M}_{I,o}^{\mathcal{P}}$  since the middle cohomology group grows much faster than all the others (although in general  $h$  is certainly a better measure of the combined size

of the Betti numbers than  $\chi$ ). It is reasonable to expect that the methods of this paper can be adapted to some PEL Shimura varieties.

**1.4. Outline of the proof.** The proof of Theorem 1.2 consists of three parts, which are more-or-less independent of each other.

In Section 3 we recall the definition of moduli schemes of  $\mathcal{D}$ -elliptic sheaves and discuss the Stein factorization of these schemes. We prove that  $\bar{M}_{I,o}^{\mathcal{D}}$  is smooth and geometrically irreducible when  $I$  is admissible.

In Section 5 we discuss  $\mathcal{D}$ -elliptic sheaves over  $\bar{\mathbb{F}}_o$  and the rationality of the corresponding points on the moduli varieties. The main tool used in this section is the description of the set of isomorphism classes of  $\mathcal{D}$ -elliptic sheaves in a given isogeny class [18, §§9-10]. We consider a special class of  $\mathcal{D}$ -elliptic sheaves, called supersingular  $\mathcal{D}$ -elliptic sheaves, and show that the corresponding points on  $\bar{M}_{I,o}^{\mathcal{D}}$  are all  $\mathbb{F}_o^{(d)}$ -rational. Moreover, we prove that when  $\deg(I)$  is large enough, compared to  $d$  and  $\deg(o)$ , the supersingular points are the only  $\mathbb{F}_o^{(d)}$ -rational points on  $\bar{M}_{I,o}^{\mathcal{D}}$  (when  $d = 2$  this can be replaced by the Drinfeld-Vladut bound). We should mention that Lachaud-Tsfasman bound is essentially equivalent to the Weil-Deligne bound when applied to  $\bar{M}_{I,o}^{\mathcal{D}}$ , hence is much larger than the limit (1.3). The number of supersingular points can be computed as a volume. Overall, we get an asymptotic formula for  $\#\bar{M}_{I,o}^{\mathcal{D}}(\mathbb{F}_o^{(d)})$ .

In Section 6 we estimate how  $h(\bar{M}_{I,o}^{\mathcal{D}})$  grows as  $\deg(I)$  tends to infinity. This relies on several deep results. First, using proper base change, we transfer the problem from  $o$ -fibre to  $\infty$ -fibre. Then a result of Schneider and Stuhler about cohomology of varieties having rigid-analytic uniformization (in combination with Berkovich's theory) reduces the problem to a calculation of the dimension of a certain space of cusp forms on  $D^\times$ . Next, a theorem of Casselman and Garland reduces the calculation of this dimension to a calculation of the Euler-Poincaré characteristic of a certain discrete cocompact subgroup of  $\mathrm{PGL}_d(F_\infty)$ . Finally, Serre's theory of Euler-Poincaré measures allows us to compute the Euler-Poincaré characteristic as a volume.

## 2. NOTATION

Aside from the notation in Introduction, we will use the following:

2.1. The residue field of  $\mathcal{O}_x$  is denoted by  $\mathbb{F}_x$ , the cardinality of  $\mathbb{F}_x$  is denoted by  $q_x$ . We assume that the valuation  $v_x : F_x \rightarrow \mathbb{Z}$  is normalized by  $v_x(\varpi_x) = 1$ , where  $\varpi_x$  is a uniformizer of  $\mathcal{O}_x$ ; the norm  $|\cdot|_x$  on  $F_x$  is  $q_x^{-v_x(\cdot)}$ . We denote the adèle ring of  $F$  by  $\mathbb{A} := \prod'_{x \in |C|} F_x$ . For a set of places  $S$  of  $F$  we denote by  $\mathbb{A}^S := \prod'_{x \notin S} F_x$  the adèle ring outside  $S$ , and  $\mathcal{D}^S := \prod_{x \notin S} \mathcal{D}_x$ .

2.2. Let  $I \neq \emptyset$  be a closed subscheme of  $C$ , and let  $\mathcal{I}$  be the ideal sheaf of  $I$ . Denote  $\mathcal{O}_I = \mathcal{O}_C/\mathcal{I}$ ,  $\deg(I) = \dim_{\mathbb{F}_q}(\mathcal{O}_I)$ , and  $\mathcal{D}_I = \mathcal{D} \otimes_{\mathcal{O}_C} \mathcal{O}_I$ . For  $I$  is disjoint from  $S$ , let

$$K_{\mathcal{D},I}^S := \ker((\mathcal{D}^S)^\times \rightarrow \mathcal{D}_I^\times).$$

2.3. Let  $\zeta_F(s) = \prod_{x \in |C|} \zeta_x(s)$  be the zeta function of  $C$ ; here  $\zeta_x(s) = (1 - q_x^{-s})^{-1}$ . It is well-known (and is easy to prove) that

$$\zeta_F(s) = \frac{1}{(1 - q^{-s})(1 - q^{1-s})},$$

Let  $\zeta_F^S(s) = \prod_{x \notin S} \zeta_x(s)$  be the partial zeta function with respect to  $S$ .

2.4. For a scheme  $W$  over  $\mathbb{F}_q$  denote by  $\text{Frob}_W$  its Frobenius endomorphism, which is the identity on the points and the  $q$ -th power map on the functions. Denote by  $C \times W$  the fibred product  $C \times_{\text{Spec}(\mathbb{F}_q)} W$ . For a sheaf  $\mathcal{F}$  on  $C$  and a sheaf  $\mathcal{G}$  on  $W$ , the sheaf  $\text{pr}_1^*(\mathcal{F}) \otimes \text{pr}_2^*(\mathcal{G})$  is denoted by  $\mathcal{F} \boxtimes \mathcal{G}$ .

2.5. Let  $G$  be the algebraic group over  $F$  defined by  $G(B) = (D \otimes_F B)^\times$  for any  $F$ -algebra  $B$ ; this is the multiplicative group of  $D$ .

### 3. MODULI SCHEMES OF $\mathcal{D}$ -ELLIPTIC SHEAVES

Let  $S$  be a  $\mathbb{F}_q$ -scheme. A  $\mathcal{D}$ -elliptic sheaf over  $S$  consists of a commutative diagram

$$(3.1) \quad \begin{array}{ccccccc} \cdots & \xrightarrow{j_{i-2}} & \mathcal{E}_{i-1} & \xrightarrow{j_{i-1}} & \mathcal{E}_i & \xrightarrow{j_i} & \mathcal{E}_{i+1} \xrightarrow{j_{i+1}} \cdots \\ & \nearrow t_{i-2} & & \nearrow t_{i-1} & & \nearrow t_i & & \nearrow t_{i+1} \\ \cdots & \xrightarrow{\tau_{j_{i-2}}} & \tau \mathcal{E}_{i-1} & \xrightarrow{\tau_{j_{i-1}}} & \tau \mathcal{E}_i & \xrightarrow{\tau_{j_i}} & \tau \mathcal{E}_{i+1} \xrightarrow{\tau_{j_{i+1}}} \cdots \end{array}$$

where each  $\mathcal{E}_i$  is a locally free  $\mathcal{O}_{C \times S}$ -module of rank  $d^2$  equipped with a right action of  $\mathcal{D}$  compatible with the  $\mathcal{O}_C$ -action,

$$\tau \mathcal{E}_i = (\text{id}_C \times \text{Frob}_S)^* \mathcal{E}_i,$$

and all  $j$ 's and  $t$ 's are  $\mathcal{O}_{C \times S}$ -linear injections compatible with the action of  $\mathcal{D}$ . All that is given is subject to the following conditions:

(i) Periodicity:

$$\mathcal{E}_{i+d} = \mathcal{E}_i \otimes_{\mathcal{O}_{C \times S}} (\mathcal{O}_C(\infty) \boxtimes \mathcal{O}_S).$$

Here  $\mathcal{E}_i$  is considered as a submodule of  $\mathcal{E}_{i+d}$  under the  $d$ -fold composition of  $j$ , and  $\mathcal{O}_C(\infty)$  is an  $\mathcal{O}_C$ -module via the natural injection  $\mathcal{O}_C \hookrightarrow \mathcal{O}_C(\infty)$ .

(ii) Pole:  $\mathcal{E}_i / j_{i-1}(\mathcal{E}_{i-1})$  is isomorphic to the direct image  $(\pi_\infty)_* \mathcal{G}_i$  of a locally free rank- $d$   $\mathcal{O}_S$ -module  $\mathcal{G}_i$  by the  $\infty$  section:

$$\pi_\infty : S \rightarrow C \times S, \quad s \mapsto (\infty, s).$$

(iii) Zero:  $\mathcal{E}_i / t_{i-1}(\tau \mathcal{E}_{i-1})$  is isomorphic to the direct image  $(\pi_z)_* \mathcal{H}_i$  of a locally free rank- $d$   $\mathcal{O}_S$ -module  $\mathcal{H}_i$  by the section

$$\pi_z : S \rightarrow C \times S, \quad s \mapsto (z(s), s),$$

where  $z : S \rightarrow C$  is a morphism of  $\mathbb{F}_q$ -schemes such that  $z(S) \subset C - R - \{\infty\}$ .

Let  $I \neq \emptyset$  be a closed subscheme of  $C - R - \{\infty\}$ . Let  $(\mathcal{E}_i, j_i, t_i)$  be a  $\mathcal{D}$ -elliptic sheaf over  $S$  such that  $z(S)$  is disjoint from  $I$ . The restriction  $\mathcal{E}_I := \mathcal{E}_i|_{I \times S}$  is independent of  $i$ , and  $t$  induces an isomorphism  $\tau \mathcal{E}_I \cong \mathcal{E}_I$ . A *level- $I$  structure* on

$(\mathcal{E}_i, j_i, t_i)$  is an  $\mathcal{O}_{I \times S}$ -linear isomorphism  $\iota : \mathcal{D}_I \boxtimes \mathcal{O}_S \cong \mathcal{E}_I$ , compatible with the action of  $\mathcal{D}_I$ , which makes the following diagram commutative:

$$\begin{array}{ccc} \tau \mathcal{E}_I & \xrightarrow{t} & \mathcal{E}_I \\ & \swarrow \tau_\iota \quad \searrow \iota & \\ & \mathcal{D}_I \boxtimes \mathcal{O}_S & \end{array}$$

For a scheme

$$z : S \rightarrow C' := C - I - R - \{\infty\},$$

denote by  $\mathbf{Ell}_I^{\mathcal{D}}(S)$  the set of isomorphism classes of  $\mathcal{D}$ -elliptic sheaves over  $S$  with level- $I$  structures. There are natural commuting actions of  $\mathbb{Z}$  and  $\mathcal{D}_I^\times$  on  $\mathbf{Ell}_I^{\mathcal{D}}(S)$ :  $n \in \mathbb{Z}$  acts by

$$[n](\mathcal{E}_i, j_i, t_i; \iota) = (\mathcal{E}_{i+n}, j_{i+n}, t_{i+n}; \iota)$$

and  $g \in \mathcal{D}_I^\times$  acts by

$$(\mathcal{E}_i, j_i, t_i; \iota)g = (\mathcal{E}_i, j_i, t_i; \iota \circ g),$$

where  $g$  acts on  $\mathcal{D}_I \boxtimes \mathcal{O}_S$  via right multiplication on  $\mathcal{D}_I$ .

By (4.1), (5.1) and (6.2) in [18], the functor  $S \mapsto \mathbf{Ell}_I^{\mathcal{D}}(S)/\mathbb{Z}$  is representable by a smooth, projective scheme  $M_I^{\mathcal{D}}$  over  $C'$  of pure relative dimension  $(d-1)$ . The action of  $\mathcal{D}_I^\times$  on  $\mathbf{Ell}_I^{\mathcal{D}}$  induces an action of this finite group on  $M_I^{\mathcal{D}}$ . We denote by  $\bar{M}_I^{\mathcal{D}}$  the quotient of  $M_I^{\mathcal{D}}$  under the action of the center  $Z(\mathcal{D}_I^\times) \cong \mathcal{O}_I^\times$  of  $\mathcal{D}_I^\times$ .

Note that if we put  $d=1$  and apply the previous definitions with  $\mathcal{D} = \mathcal{O}_C$ , then we arrive at the notion of  $\mathcal{O}_C$ -elliptic sheaves with level- $I$  structures. This case was considered much earlier by Drinfeld [7], who proved that  $M_I^{\mathcal{O}_C}$  is isomorphic to the moduli scheme of rank-1 Drinfeld  $A$ -modules with level- $I$  structures. This latter moduli scheme is closely related to Class Field Theory of  $F$ ; see [6, Thm. 1].

As follows from [16, pp. 26-29], there is a natural morphism of schemes over  $C'$

$$\wp : M_I^{\mathcal{D}} \rightarrow M_I^{\mathcal{O}_C}$$

which is compatible with the action of  $\mathcal{D}_I^\times$  in the sense that  $\wp \circ g = \det(g) \circ \wp$  for any  $g \in \mathcal{D}_I^\times \cong \mathrm{GL}_d(\mathcal{O}_I)$ .

**Proposition 3.1.** *The fibres of  $\wp$  are geometrically irreducible.*

*Proof.* By Stein factorization theorem, it is enough to show that the fibres of

$$\wp_{\bar{\eta}} : M_{I, \bar{\eta}}^{\mathcal{D}} := M_I^{\mathcal{D}} \times_{C'} \mathrm{Spec}(\bar{F}) \rightarrow M_{I, \bar{\eta}}^{\mathcal{O}_C} := M_I^{\mathcal{O}_C} \times_{C'} \mathrm{Spec}(\bar{F})$$

are connected, where  $\bar{F}$  denotes a fixed algebraic closure of  $F$ . Since  $\wp_{\bar{\eta}}$  is  $\mathcal{D}_I^\times$ -equivariant and  $\det : \mathcal{D}_I^\times \rightarrow \mathcal{O}_I^\times$  is surjective,  $\wp_{\bar{\eta}}$  is surjective. Hence it is enough to show that the number of connected components of  $M_{I, \bar{\eta}}^{\mathcal{D}}$  and  $M_{I, \bar{\eta}}^{\mathcal{O}_C}$  are the same.

By Class Field Theory,  $M_{I, \bar{\eta}}^{\mathcal{O}_C}$  is a disjoint union of  $\# [F^\times \setminus (\mathbb{A}^\infty)^\times / K_{\mathcal{O}_C, I}^\infty]$  copies of  $\mathrm{Spec}(\bar{F})$  and  $\mathcal{O}_I^\times$  acts transitively on these points. On the other hand, the number of connected components of  $M_{I, \bar{\eta}}^{\mathcal{D}}$  is equal to the 0-th Betti number  $h_{I, \bar{\eta}}^0$  (in the notation of Section 6), so the desired claim follows from Corollary 6.2.  $\square$

**Corollary 3.2.** *If  $I = \mathrm{Spec}(A/\mathfrak{p})$  is admissible then  $\bar{M}_{I, o}^{\mathcal{D}}$  is a smooth, projective, geometrically irreducible,  $(d-1)$ -dimensional variety defined over  $\mathbb{F}_o$ , which is a form of one of the components of  $M_{I, o}^{\mathcal{D}}$ .*

*Proof.* The group  $\mathbb{F}_{\mathfrak{p}}^{\times}/\mathbb{F}_q^{\times}$  acts freely and transitively on the geometrically irreducible components of  $M_{I,o}^{\mathcal{D}_C}$ . On the other hand, it is easy to see that the action of  $Z(\mathcal{D}_I^{\times}) \cong \mathbb{F}_{\mathfrak{p}}^{\times}$  on  $M_{I,o}^{\mathcal{D}}$  factors through  $\mathbb{F}_{\mathfrak{p}}^{\times}/\mathbb{F}_q^{\times}$ . By Proposition 3.1,  $\wp_o \circ z = z^d \circ \wp_o$  for  $z \in \mathbb{F}_{\mathfrak{p}}^{\times}$ . Since  $\mathfrak{p}$  is admissible,  $(\mathbb{F}_{\mathfrak{p}}^{\times})^d$  surjects onto  $\mathbb{F}_{\mathfrak{p}}^{\times}/\mathbb{F}_q^{\times}$ . We conclude that  $\mathbb{F}_{\mathfrak{p}}^{\times}/\mathbb{F}_q^{\times}$  acts freely and transitively also on the geometrically irreducible components of  $M_{I,o}^{\mathcal{D}}$ . This implies the claim.  $\square$

#### 4. VOLUME CALCULATION

This section is of auxiliary nature. Here we compute a certain volume which is used in Sections 5 and 6. This result should be well-known, but in absence of a convenient explicit reference we sketch some of the details.

For  $x \in |C|$ , normalize the Haar measure  $dg_x$  on  $G(F_x)$  by  $\text{Vol}(\mathcal{D}_x^{\times}, dg_x) = 1$ . Fix the Haar measure  $d\bar{g}$  on  $G(\mathbb{A})$  to be the restricted product measure. This measure will be called the *canonical product measure*.

Consider the homomorphism

$$(4.1) \quad \|\cdot\| : G(\mathbb{A}) \rightarrow q^{\mathbb{Z}}$$

given by the composition of the reduced norm  $\text{Nr} : G(\mathbb{A}) \rightarrow \mathbb{A}^{\times}$  with the idelic norm  $\prod_{x \in |C|} |\cdot|_x : \mathbb{A}^{\times} \rightarrow q^{\mathbb{Z}}$ . Denote the kernel of this homomorphism by  $G^1(\mathbb{A})$ . The group  $G(F)$ , under the diagonal embedding into  $G(\mathbb{A})$ , lies in  $G^1(\mathbb{A})$ , thanks to the product formula. The quotient  $G(F) \backslash G^1(\mathbb{A})$  is compact, hence has finite volume. The main result of this section is the following:

**Proposition 4.1.**

$$\text{Vol}(G(F) \backslash G^1(\mathbb{A}), d\bar{g}) = \frac{1}{(q-1)} \prod_{i=1}^{d-1} \zeta_F^R(-i).$$

*Proof.* Let  $\Phi$  be the characteristic function of  $\mathcal{D}$  in  $D(\mathbb{A})$ . Consider the following integral

$$\zeta_D(s) = \int_{G(\mathbb{A})} \Phi(g) \|g\|^s d\bar{g}.$$

It absolutely converges for  $\text{Re}(s) > 1$ , can be meromorphically continued to the whole plane with a simple pole at 0, cf. [33, §3.1]. Moreover, from the calculations in *loc.cit.*, one can deduce that

$$(4.2) \quad \text{Res}_{s=0} \zeta_D(s) = -\text{Vol}(G(F) \backslash G^1(\mathbb{A}), d\bar{g}) \frac{1}{\log q}.$$

Next, we have the decompositions  $\Phi = \prod_x \Phi_x$  and  $\zeta_D(s) = \prod_x \zeta_{D_x}(s)$ , where  $\Phi_x$  is the characteristic function of  $\mathcal{D}_x$  in  $D_x$  and

$$\zeta_{D_x}(s) = \int_{G(F_x)} \Phi_x(g_x) \cdot |\text{Nr}(g_x)|_x^s dg_x.$$



If  $D$  is split at  $x$  then, considering the decomposition of  $\mathbb{M}_d(\mathcal{O}_x)$  into left  $\mathcal{D}_x^\times \cong \mathrm{GL}_d(\mathcal{O}_x)$ -cosets, we get

$$\begin{aligned}\zeta_{D_x}(s) &= \int_{\mathbb{M}_d(\mathcal{O}_x) - \{0\}} |\mathrm{Nr}(g_x)|_x^s dg_x \\ &= \mathrm{Vol}(\mathcal{D}_x^\times) \sum_{(n_i) \in \mathbb{Z}_{\geq 0}^d} q_x^{(-sn_1 + (-s+1)n_2 + \cdots + (-s+d-1)n_d)} \\ &= \zeta_x(s) \cdot \zeta_x(s-1) \cdots \zeta_x(s-(d-1)).\end{aligned}$$

On the other hand, if  $D$  is ramified at  $x$  then  $D_x$  is a division algebra by assumption, hence  $\mathcal{D}_x$  has a unique maximal ideal  $\mathfrak{P} \triangleleft \mathcal{D}_x$  and  $\mathrm{Nr}(\mathfrak{P}) = \varpi_x$ ; see [23, Thm. 24.13]. Thus,

$$\zeta_{D_x}(s) = \int_{\mathcal{D}_x - \{0\}} |\mathrm{Nr}(g_x)|_x^s dg_x = \mathrm{Vol}(\mathcal{D}_x^\times) \sum_{n \in \mathbb{Z}_{\geq 0}} q_x^{-sn} = \zeta_x(s).$$

Combining these local calculations,

$$(4.3) \quad \mathrm{Res}_{s=0} \zeta_D(s) = -\frac{1}{(q-1)\log q} \prod_{i=1}^{d-1} \zeta_F^R(-i).$$

From (4.2) and (4.3) we deduce

$$\mathrm{Vol}(G(F) \backslash G^1(\mathbb{A}), d\bar{g}) = \frac{1}{(q-1)} \prod_{i=1}^{d-1} \zeta_F^R(-i),$$

as was required.  $\square$

## 5. SUPERSINGULAR $\mathcal{D}$ -ELLIPTIC SHEAVES

In this section we discuss  $\mathcal{D}$ -elliptic sheaves over  $k := \overline{\mathbb{F}}_o$  and the rationality of the corresponding points on the moduli schemes.

**5.1. Isogeny classes.** One of the key preliminary results in [18] is the description of the points on the closed fibres of  $M_I^{\mathcal{D}} \rightarrow C'$ . This is done in two steps, similar to the description of the set of abelian varieties over finite fields: one starts by describing the isogeny classes of  $\mathcal{D}$ -elliptic sheaves over  $k$  (as in Honda-Tate theory) and then parametrizes  $\mathcal{D}$ -elliptic sheaves in each isogeny class. We start by recalling this description.

**Definition 5.1.** ([18, (9.11)]) A  $(D, \infty, o)$ -type is a pair  $(\tilde{F}, \tilde{\Pi})$ , where  $\tilde{F}$  is a finite separable field extension of  $F$  and  $\tilde{\Pi} \in \tilde{F}^\times \otimes_{\mathbb{Z}} \mathbb{Q}$ , satisfying the following conditions:

- For a proper subfield  $\tilde{F}' \subsetneq \tilde{F}$ ,  $\tilde{\Pi} \notin (\tilde{F}')^\times \otimes_{\mathbb{Z}} \mathbb{Q}$ .
- $[\tilde{F} : F]$  divides  $d$ .
- $F_\infty \otimes_F \tilde{F}$  is a field and, if  $\infty$  is the unique place of  $\tilde{F}$  which divides  $\infty$ , we have

$$\deg(\infty) \cdot v_\infty(\tilde{\Pi}) = -[\tilde{F} : F]/d.$$

- There exists a unique place  $\tilde{o} \neq \infty$  of  $\tilde{F}$  such that  $v_{\tilde{o}}(\tilde{\Pi}) \neq 0$ ; moreover  $\tilde{o}$  divides  $o$ .
- For each place  $x$  of  $F$  and each  $\tilde{x}$  of  $\tilde{F}$  dividing  $x$ , we have

$$(d[\tilde{F}_{\tilde{x}} : F_x]/[\tilde{F} : F]) \cdot \mathrm{inv}_x(D) \in \mathbb{Z}.$$

In [18, (9.2)] the authors introduce the notion of isogenies between  $\mathcal{D}$ -elliptic sheaves over  $k$ . We will not recall this definition; for our purposes it is enough to know [18, (9.13)] that there is a canonical bijection between the set of isogeny classes of  $\mathcal{D}$ -elliptic sheaves over  $k$  and the set of isomorphism classes of  $(D, \infty, o)$ -types.

Assume  $I \subset C - R - \{o, \infty\}$  and denote by  $M_{I,o}^{\mathcal{D}} := M_I^{\mathcal{D}} \times_{C'} \text{Spec}(\mathbb{F}_o)$  the fibre of  $M_I^{\mathcal{D}}$  over  $o$ . Fix a  $(D, \infty, o)$ -type  $(\tilde{F}, \tilde{\Pi})$  and denote by

$$M_{I,o}^{\mathcal{D}}(k)_{(\tilde{F}, \tilde{\Pi})} \subset M_{I,o}^{\mathcal{D}}(k)$$

the set of isomorphism classes of  $\mathcal{D}$ -elliptic sheaves over  $k$  which are in the isogeny class corresponding to  $(\tilde{F}, \tilde{\Pi})$ .

Let  $\Delta$  be the central division algebra over  $\tilde{F}$  with invariants

$$\text{inv}_{\tilde{x}} \Delta = \begin{cases} [\tilde{F} : F]/d, & \text{if } \tilde{x} = \tilde{\infty}; \\ -[\tilde{F} : F]/d, & \text{if } \tilde{x} = \tilde{o}; \\ [\tilde{F}_{\tilde{x}} : F_x] \cdot \text{inv}_x(D), & \text{otherwise.} \end{cases}$$

Let

$$h = [\tilde{F}_{\tilde{o}} : F_o]d/[\tilde{F} : F].$$

$\Delta^\times$  naturally acts on the Dieudonné modules of a  $\mathcal{D}$ -elliptic sheaf in the isogeny class of  $(\tilde{F}, \tilde{\Pi})$ , and these actions induce group homomorphisms

$$\begin{cases} \Delta^\times \hookrightarrow G(\mathbb{A}^{\infty, o}) \\ \Delta^\times \hookrightarrow \text{GL}_{d-h}(F_o) \\ \Delta^\times \hookrightarrow N_{o,h}^\times \xrightarrow{\text{Nr}} F_o^\times \xrightarrow{v_o} \mathbb{Z}, \end{cases}$$

where  $N_{o,h}$  is the central division algebra over  $F_o$  with invariant  $-1/h$ ; see [18, p. 270]. The main result of [18, §10] is the following:

**Theorem 5.2.** *There is a bijection*

$$M_{I,o}^{\mathcal{D}}(k)_{(\tilde{F}, \tilde{\Pi})} \xrightarrow{\sim} \Delta^\times \setminus (Y_I^{\infty, o} \times Y_o^{\tilde{o}} \times Y_{\tilde{o}}),$$

compatible with the action of  $\mathcal{D}_I^\times$ , where

$$Y_I^{\infty, o} := G(\mathbb{A}^{\infty, o})/K_{\mathcal{D}, I}^{\infty, o}, \quad Y_o^{\tilde{o}} := \text{GL}_{d-h}(F_o)/\text{GL}_{d-h}(\mathcal{O}_o), \quad Y_{\tilde{o}} := \mathbb{Z}.$$

The action of  $\text{Frob}_o$  on  $M_{I,o}^{\mathcal{D}}(k)_{(\tilde{F}, \tilde{\Pi})}$  corresponds to translation by 1 on  $Y_{\tilde{o}} = \mathbb{Z}$ .

**5.2. Rational points.** We say that a  $\mathcal{D}$ -elliptic sheaf over  $k$  is *supersingular* if in its  $(D, \infty, o)$ -type  $\tilde{F} = F$ . It is not hard to show that all supersingular  $\mathcal{D}$ -elliptic sheaves are isogenous [3, Prop. 10.2.1], i.e., there is a unique  $(D, \infty, o)$ -type  $(F, \Pi)$ .

We have an action of  $\mathcal{D}_I^\times \cong \text{GL}_d(\mathcal{O}_I)$  on  $M_{I,o}^{\mathcal{D}}$  via its action on level structures in the moduli problem. Denote the quotient of  $M_{I,o}^{\mathcal{D}}$  under the action of the center  $Z(\mathcal{D}_I^\times) \cong \mathcal{O}_I^\times$  of  $\mathcal{D}_I^\times$  by

$$\bar{M}_{I,o}^{\mathcal{D}} := M_{I,o}^{\mathcal{D}}/Z(\mathcal{D}_I^\times).$$

Denote the preimage of  $Z(\mathcal{D}_I^\times)$  in  $(\mathcal{D}^{\infty, o})^\times$  under  $(\mathcal{D}^{\infty, o})^\times \rightarrow \mathcal{D}_I^\times$  by  $\bar{K}_{\mathcal{D}, I}^{\infty, o}$ , and denote by  $\bar{M}_{I,o}^{\mathcal{D}}(k)_{(\tilde{F}, \tilde{\Pi})}$  the image of  $M_{I,o}^{\mathcal{D}}(k)_{(\tilde{F}, \tilde{\Pi})}$  under the quotient map  $M_{I,o}^{\mathcal{D}} \rightarrow \bar{M}_{I,o}^{\mathcal{D}}$ .

**Proposition 5.3.** *There is a bijection*

$$\bar{M}_{I,o}^{\mathcal{D}}(k)_{(\bar{F},\bar{\Pi})} \xrightarrow{\sim} \Delta^\times \setminus (\bar{Y}_I^{\infty,o} \times Y_o^{\tilde{o}} \times Y_{\tilde{o}}),$$

where  $\bar{Y}_I^{\infty,o} := G(\mathbb{A}^{\infty,o})/\bar{K}_{\mathcal{D},I}^{\infty,o}$ . The action of  $\text{Frob}_o$  on  $\bar{M}_{I,o}^{\mathcal{D}}(k)_{(\bar{F},\bar{\Pi})}$  corresponds to translation by 1 on  $Y_{\tilde{o}} = \mathbb{Z}$ .

*Proof.* This is a consequence of Theorem 5.2.  $\square$

From this proposition it is not hard to deduce the following criterion for the existence of  $\mathbb{F}_o^{(n)}$ -rational points on  $\bar{M}_{I,o}^{\mathcal{D}}$ , cf. [18, p. 278]:

$$P = \Delta^\times [g^{\infty,o} \bar{K}_{\mathcal{D},I}^{\infty,o} g_o^{\tilde{o}} \text{GL}_{d-h}(\mathcal{O}_o), m_{\tilde{o}}] \in \bar{M}_{I,o}^{\mathcal{D}}(k)_{(\bar{F},\bar{\Pi})}.$$

is rational over  $\mathbb{F}_o^{(n)}$  if and only if there exists  $\delta \in \Delta^\times$  such that

$$(5.1) \quad \begin{cases} (g^{\infty,o})^{-1} \delta g^{\infty,o} \in \bar{K}_{\mathcal{D},I}^{\infty,o} \\ (g_o^{\tilde{o}})^{-1} \delta g_o^{\tilde{o}} \in \text{GL}_{d-h}(\mathcal{O}_o) \\ v_{\tilde{o}}(\text{Nr}(\delta)) = n \deg(o) / \deg(\tilde{o}) \end{cases}$$

where  $\text{Nr}$  is the reduced norm on  $\Delta$ .

Let  $B$  be a finite dimensional  $K$ -algebra, where  $K$  is a field. The left multiplication by  $\alpha \in B$  gives a  $K$ -linear transformation of  $B$  (as a finite dimensional vector space over  $K$ ). Define the characteristic polynomial  $\text{ch.p.}_{B/K} \alpha \in K[X]$  of  $\alpha$  to be the characteristic polynomial of this transformation, and  $\det_{B/K}(\alpha)$  be its determinant.

**Proposition 5.4.** *Suppose  $n$  is fixed and  $I$  has in its support a prime  $\mathfrak{p} \triangleleft A$  whose degree is large enough compared to  $n$ . Then*

$$\bar{M}_{I,o}^{\mathcal{D}}(\mathbb{F}_o^{(n)}) = \bar{M}_{I,o}^{\mathcal{D}}(\mathbb{F}_o^{(n)})_{(F,\Pi)}.$$

*In other terms, the images of supersingular points on  $\bar{M}_{I,o}^{\mathcal{D}}$  are the only possible  $\mathbb{F}_o^{(n)}$ -rational points.*

*Proof.* Suppose  $P \in \bar{M}_{I,o}^{\mathcal{D}}(\mathbb{F}_o^{(n)})_{(\bar{F},\bar{\Pi})}$ . We need to show  $\tilde{F} = F$ . Let  $\delta \in \Delta^\times$  be an element corresponding to  $P$ . Let  $F' = F[\delta]$  be the field generated by  $\delta$  over  $F$  and  $f_\delta \in F[X]$  be the minimal polynomial of  $\delta$ . The conditions in (5.1) imply that  $\delta$  is integral over  $A$ , so  $f_\delta$  is a monic irreducible polynomial in  $A[X]$ .

First, assume  $F'/F$  is separable. Let  $\text{disc}(\delta) \in A$  be the discriminant of  $f_\delta$ . We can consider  $\Delta$  as a finite dimensional algebra over  $F$ . Since  $f_\delta$  divides  $\text{ch.p.}_{\Delta/F} \delta$ , the constant term  $c_0(\delta)$  of  $f_\delta$  divides  $\det_{\Delta/F}(\delta)$ . By [23, (9.12)],

$$\det_{\Delta/F}(\delta) = \text{Nr}_{\tilde{F}/F}(\det_{\Delta/\tilde{F}}(\delta)).$$

On the other hand, since  $\Delta$  is a central division algebra over  $\tilde{F}$  of dimension  $\left(\frac{d}{[F:F]}\right)^2$ ,

$$\det_{\Delta/\tilde{F}}(\delta) = \text{Nr}(\delta)^{d/[F:F]}.$$

Since  $\text{Nr}(\delta)$  has zero valuation at every finite place of  $\tilde{F}$  except at  $\tilde{o}$  where the valuation is bounded in terms of  $n$ ,  $d$  and  $\deg(o)$ , we conclude that  $\deg(c_0(\delta))$  is bounded in terms of  $n$ ,  $d$  and  $\deg(o)$ . The degree of  $f_\delta$  is bounded by  $d$ , so from the definition of discriminant,  $\deg(\text{disc}(\delta)) \leq d(d-1) \deg(c_0(\delta))$ . We conclude that  $\deg(\text{disc}(\delta))$  is bounded in terms of  $n$ ,  $d$  and  $\deg(o)$ .

On the other hand, the fact that  $\delta$  can be conjugated to lie in  $\bar{K}_{\mathcal{D},I}^{\infty,o}$  implies that  $f_\delta$  modulo  $\mathfrak{p}$  is a power of a linear polynomial. Therefore,  $\mathfrak{p}$  must divide  $\text{disc}(\delta)$ , unless  $f_\delta$  is a linear polynomial. Since the degree of the discriminant of  $\delta$  is bounded, if  $\deg(\mathfrak{p})$  is large enough compared to  $n$ , then  $f_\delta$  has to be a linear polynomial, so  $\delta \in F$  and  $F' = F$ . Finally, by [17, Lem. 3.3.6],  $\tilde{F} \subset F'$ . Thus,  $\tilde{F} = F$  as we wanted to show.

Now assume  $F'/F$  is inseparable. There is a minimal  $m \geq 1$  such that  $\delta' = \delta^{p^m}$  is separable over  $F$ . Since  $F' \subset \Delta$ , the power  $p^m$  is bounded in terms of  $d$ . It is clear that if  $\delta$  satisfies the conditions in (5.1), then  $\delta'$  satisfies similar conditions with  $n$  replaced by  $n' := n \cdot p^m$ . It is also clear that if  $P \in \bar{M}_{I,o}^{\mathcal{D}}(\mathbb{F}_o^{(n)})_{(\tilde{F},\tilde{\Pi})}$  then  $P \in \bar{M}_{I,o}^{\mathcal{D}}(\mathbb{F}_o^{(n')})_{(\tilde{F},\tilde{\Pi})}$  and  $\delta'$  is its corresponding element of  $\Delta^\times$ . This reduced the situation to the initial case and finishes the proof.  $\square$

**Proposition 5.5.** *The images of supersingular points are rational over  $\mathbb{F}_o^{(d)}$  :*

$$\bar{M}_{I,o}^{\mathcal{D}}(\mathbb{F}_o^{(d)})_{(F,\Pi)} = \bar{M}_{I,o}^{\mathcal{D}}(k)_{(F,\Pi)}.$$

*Proof.* Since  $\tilde{F} = F$ ,  $h = d$  and there is a unique  $\tilde{o}$  dividing  $o$ , namely  $o$  itself. Hence a point  $P$  on  $M_{I,o}(k)_{(F,\Pi)}$  is given by  $\Delta^\times[g^{\infty,o}\bar{K}_{\mathcal{D},I}^{\infty,o}, m_o]$ , and  $P$  is rational over  $\mathbb{F}_o^{(d)}$  if there is  $\delta \in \Delta^\times$  such that

$$\begin{cases} (g^{\infty,o})^{-1}\delta g^{\infty,o} \in \bar{K}_{\mathcal{D},I}^{\infty,o} \\ v_o(\text{Nr}(\delta)) = d. \end{cases}$$

Let  $f \in F^\times$  be an element whose divisor on  $C$  is  $(f) = [o - \deg(o)\infty]$ . Take  $\delta = f \in F^\times \hookrightarrow \Delta^\times$ , as an element of the center of  $\Delta^\times$ . Then  $\text{Nr}(f) = f^d$ , so  $v_o(\text{Nr}(f)) = d$ . We clearly have  $f \in \bar{K}_{\mathcal{D},I}^{\infty,o}$ , so  $(g^{\infty,o})^{-1}fg^{\infty,o} = f \in \bar{K}_{\mathcal{D},I}^{\infty,o}$ , and the first condition is also satisfied. (Note that  $f$  usually will not be in  $K_{\mathcal{D},I}^{\infty,o}$ .)  $\square$

**Theorem 5.6.** *Let  $S = R \cup \{\infty, o\}$ . Let  $I = \text{Spec}(A/\mathfrak{p}) \neq o$  be admissible. For all but finitely many  $I$ ,*

$$\#\bar{M}_{I,o}^{\mathcal{D}}(\mathbb{F}_o^{(d)}) = \#\text{PGL}_d(\mathbb{F}_{\mathfrak{p}}) \cdot \prod_{i=1}^{d-1} \zeta_F^S(-i).$$

*Proof.* Let  $\bar{D}$  be the  $d^2$ -dimensional central division algebra over  $F$  whose invariants are  $\text{inv}_o(\bar{D}) = -1/d$ ,  $\text{inv}_\infty(\bar{D}) = 1/d$ , and  $\text{inv}_x(\bar{D}) = \text{inv}_x(D)$  if  $x \neq o, \infty$ . In particular,  $\bar{D}_x$  is a division algebra for every  $x \in S$ ; cf. [23, §32]. Let  $\bar{\mathcal{D}}$  be a locally free sheaf of  $\mathcal{O}_C$ -algebras with generic fibre  $\bar{D}$ , and such that  $\bar{\mathcal{D}}_x$  is a maximal order in  $\bar{D}_x$  for any  $x \in |C|$ . Let  $\bar{G}$  be the group of units in  $\bar{D}$ . Let  $K_{\bar{\mathcal{D}},I} := \ker(\bar{\mathcal{D}}^\times \rightarrow \bar{\mathcal{D}}_I^\times)$ , and let  $K_{\bar{\mathcal{D}},I}^{\infty,o}$  and  $K_{\bar{\mathcal{D}},I}^\infty$  be defined similarly to §2.2.

By Theorem 5.2,

$$(5.2) \quad \#M_{I,o}^{\mathcal{D}}(k)_{(F,\Pi)} = \#(\bar{G}(F) \setminus \bar{G}(\mathbb{A}^{\infty,o})/K_{\bar{\mathcal{D}},I}^{\infty,o} \times \mathbb{Z}).$$

(Note that  $G(\mathbb{A}^{\infty,o}) \cong \bar{G}(\mathbb{A}^{\infty,o})$  and  $K_{\bar{\mathcal{D}},I}^{\infty,o} \cong K_{\bar{\mathcal{D}},I}^{\infty,o}$ ). Since  $\bar{D}_o$  is a division algebra,  $v_o \circ \text{Nr} : G(F_o)/\bar{\mathcal{D}}_o^\times \cong \mathbb{Z}$ ; see [23, §12]. But  $\bar{G}(F)$  acts on  $\mathbb{Z}$  in (5.2) via  $v_o \circ \text{Nr}$ , so

$$\bar{G}(F) \setminus \bar{G}(\mathbb{A}^{\infty,o})/K_{\bar{\mathcal{D}},I}^{\infty,o} \times \mathbb{Z} \cong \bar{G}(F) \setminus \bar{G}(\mathbb{A}^\infty)/K_{\bar{\mathcal{D}},I}^\infty.$$

Since  $\bar{D}_\infty$  is a division algebra and  $\infty$  is rational, for each  $a \in \bar{G}(\mathbb{A}^\infty)$ , up to an element of  $\bar{\mathcal{D}}_\infty^\times$ , there is a unique  $b \in \bar{G}(F_\infty)$  such that  $(a, b) \in \bar{G}^1(\mathbb{A})$ , cf. [23, §13].

Hence

$$\bar{G}(F) \setminus \bar{G}(\mathbb{A}^\infty)/K_{\bar{\mathcal{D}},I}^\infty \cong \bar{G}(F) \setminus \bar{G}^1(\mathbb{A})/K_{\bar{\mathcal{D}},I}.$$

If  $I \neq \emptyset$  then  $K_{\bar{\mathcal{D}},I} \cap \bar{G}(F) = 1$ , since only the constants are everywhere integral but the only constant which reduces to the identity modulo  $I$  is 1. Also note that  $\bar{\mathcal{D}}^\times/K_{\bar{\mathcal{D}},I} \cong \mathrm{GL}_d(\mathcal{O}_I)$ , so with respect to the canonical product measure on  $\bar{G}(\mathbb{A})$  we have

$$\#(\bar{G}(F) \setminus \bar{G}^1(\mathbb{A})/K_{\bar{\mathcal{D}},I}) = \#\mathrm{GL}_d(\mathcal{O}_I) \cdot \mathrm{Vol}(\bar{G}(F) \setminus \bar{G}^1(\mathbb{A}), d\bar{g}).$$

Finally, using Proposition 4.1,

$$\#M_{I,o}^{\mathcal{P}}(k)_{(F,\Pi)} = \frac{\#\mathrm{GL}_d(\mathcal{O}_I)}{(q-1)} \prod_{i=1}^{d-1} \zeta_F^S(-i).$$

Since  $\mathbb{F}_{\mathfrak{p}}^\times/\mathbb{F}_q^\times$  acts freely and transitively on the components of  $M_{I,o}^{\mathcal{P}}$  and preserves  $M_{I,o}^{\mathcal{P}}(k)_{(F,\Pi)}$ ,

$$\#\bar{M}_{I,o}^{\mathcal{P}}(k)_{(F,\Pi)} = \#\mathrm{PGL}_d(\mathbb{F}_{\mathfrak{p}}) \cdot \prod_{i=1}^{d-1} \zeta_F^S(-i).$$

Now the theorem follows from Propositions 5.4 and 5.5.  $\square$

## 6. ASYMPTOTIC BETTI NUMBERS

Fix a prime  $\ell$  not equal to the characteristic  $p$  of  $F$  and consider the  $\ell$ -adic cohomology groups

$$H_{I,o}^i := H^i(M_{I,o}^{\mathcal{P}} \otimes_{\mathbb{F}_o} k, \bar{\mathbb{Q}}_\ell), \quad i \geq 0.$$

Each  $H_{I,o}^i$  is a finite dimensional  $\bar{\mathbb{Q}}_\ell$ -vector space, which is 0 for  $i \geq 2(d-1)$ . Denote  $h_{I,o}^i = \dim_{\bar{\mathbb{Q}}_\ell} H_{I,o}^i$  and  $h_{I,o} = \sum_{i=0}^{2d-2} h_{I,o}^i$ . In this section we estimate how  $h_{I,o}$  grows as  $\deg(I)$  tends to infinity.

Let  $\eta = \mathrm{Spec}(F)$  be the generic point of  $C$ . We also have the  $\ell$ -adic cohomology groups of the generic fibre  $M_{I,\eta}^{\mathcal{P}}$  of  $M_I^{\mathcal{P}}$ :

$$H_{I,\eta}^i := H^i(M_{I,\eta}^{\mathcal{P}} \otimes_F \bar{F}, \bar{\mathbb{Q}}_\ell), \quad i \geq 0.$$

By the proper base change theorem, for all  $i \geq 0$  there is a canonical isomorphism of  $\bar{\mathbb{Q}}_\ell$ -vector spaces  $H_{I,\eta}^i \cong H_{I,o}^i$ . Hence we concentrate on estimating  $h_I = \sum_{i=0}^{2d-2} h_{I,\eta}^i$ , where  $h_{I,\eta}^i = \dim_{\bar{\mathbb{Q}}_\ell} H_{I,\eta}^i$ .

For two  $\mathbb{Q}$ -valued functions  $f(I)$  and  $g(I)$  depending on  $I$ , we write  $f(I) \sim g(I)$  if  $f(I)/g(I) \rightarrow 1$  as  $\deg(I) \rightarrow \infty$ .

**6.1. Cohomology and automorphic representations.** Let  $\mathcal{A}$  be the space of locally constant  $\bar{\mathbb{Q}}_\ell$ -valued functions on the double coset space  $G(F) \setminus G(\mathbb{A})/\varpi_\infty^\mathbb{Z}$ . This space is equipped with the right regular representation of  $G(\mathbb{A})$ . Since  $D$  is a division algebra, the coset space  $G(F) \setminus G(\mathbb{A})/\varpi_\infty^\mathbb{Z}$  is compact, so  $\mathcal{A}$  decomposes as a direct sum of irreducible admissible representations  $\Pi$  of  $G(\mathbb{A})$  with finite multiplicities  $m(\Pi) \geq 0$ , cf. [18, p. 291]:

$$(6.1) \quad \mathcal{A} = \bigoplus_{\Pi} m(\Pi) \cdot \Pi.$$

We will refer to the representations appearing in this sum with non-zero multiplicities as the *automorphic representations* of  $G(\mathbb{A})$ . Each automorphic representation

$\Pi$  decomposes as a restricted tensor product  $\bigotimes_{x \in |C|} \Pi_x$  of irreducible admissible representations of  $G(F_x)$ . Denote  $\Pi^\infty := \bigotimes_{x \neq \infty} \Pi_x$ , so  $\Pi = \Pi^\infty \otimes \Pi_\infty$ .

Among the automorphic representations of  $G(\mathbb{A})$  we have the *characters*  $\chi \circ \text{Nr}$ , where  $\text{Nr} : G(\mathbb{A}) \rightarrow \mathbb{A}^\times$  is the reduced norm and  $\chi$  is a Hecke character on  $\mathbb{A}^\times$  with  $\chi_\infty = 1$ . These representations are clearly 1-dimensional and it is known that they appear with multiplicity 1 in the decomposition (6.1). Any other automorphic representation is infinite dimensional.

The *Steinberg representation*  $\text{St}_\infty$  of  $G(F_\infty) \cong \text{GL}_d(F_\infty)$  is the unique irreducible quotient of the induced representation

$$\text{Ind}_{B(F_\infty)}^{\text{GL}_d(F_\infty)} \left( |\cdot|_\infty^{-\frac{d-1}{2}}, |\cdot|_\infty^{-\frac{d-3}{2}}, \dots, |\cdot|_\infty^{-\frac{d-3}{2}}, |\cdot|_\infty^{-\frac{d-1}{2}} \right),$$

where  $B \subset \text{GL}_d$  is the standard Borel subgroup of the upper triangular matrices.

Let  $\mathbb{T}_I$  be the Hecke algebra of  $\mathbb{Q}$ -valued locally constant functions with compact supports on  $G(\mathbb{A}^\infty)$  invariant under the left and right translation by  $K_{\mathcal{D},I}^\infty$ . (The product on  $\mathbb{T}_I$  is given by the convolution with respect to the Haar measure on  $G(\mathbb{A}^\infty)$  which gives the volume 1 to the open compact subgroup  $K_{\mathcal{D},I}^\infty \subset G(\mathbb{A}^\infty)$ .) There is a natural action of  $\mathbb{T}_I$  on each  $H_{I,\eta}^i$  which commutes with the action of  $\text{Gal}(\bar{F}/F)$ , cf. [18, §12]. Denote by  $(H_{I,\eta}^i)^{\text{ss}}$  the associated semi-simplification of  $H_{I,\eta}^i$  as a  $\text{Gal}(\bar{F}/F) \times \mathbb{T}_I$ -module. One of the main results in [18] relates  $(H_{I,\eta}^i)^{\text{ss}}$  to automorphic representations. Taking the  $K_{\mathcal{D},I}^\infty$ -invariants in (14.9), (14.10) and (14.12) of *loc.cit.*, one obtains:

**Theorem 6.1.**

$$(H_{I,\eta}^i)^{\text{ss}} = \bigoplus_{\Pi} V_{\Pi}^i \otimes (\Pi^\infty)^{K_{\mathcal{D},I}^\infty},$$

where the sum is over automorphic representations of  $G(\mathbb{A})$  such that either  $\Pi_\infty = \mathbf{1}$  is the trivial character or  $\Pi_\infty = \text{St}_\infty$ , and  $V_{\Pi}^i$  is a finite dimensional  $\overline{\mathbb{Q}}_\ell$  representation of  $\text{Gal}(\bar{F}/F)$ .

If  $\Pi_\infty = \mathbf{1}$  then  $\Pi = \chi \circ \text{Nr}$  for a Hecke character  $\chi$  on  $\mathbb{A}^\times$ . In this case,  $V_{\Pi}^0$  is the Galois character corresponding to  $\chi$  by Class Field Theory,  $V_{\Pi}^{2i}$  is isomorphic to the Tate twist  $V_{\Pi}^0(-i)$  and  $V_{\Pi}^{2i+1} = 0$ ,  $0 \leq i \leq d-1$ .

If  $\Pi_\infty = \text{St}_\infty$  then  $V_{\Pi}^i = 0$  for  $i \neq d-1$  and  $\dim_{\overline{\mathbb{Q}}_\ell} V_{\Pi}^{d-1} = m(\Pi) \cdot d$ .

**Corollary 6.2.**  $h_{I,\eta}^0 = \# [F^\times \setminus (\mathbb{A}^\infty)^\times / K_{\mathcal{O},I}^\infty]$ .

*Proof.* By the strong approximation theorem [22], the reduced norm induces an isomorphism

$$G(F) \setminus G(\mathbb{A}^\infty) / K_{\mathcal{D},I}^\infty \cong F^\times \setminus (\mathbb{A}^\infty)^\times / K_{\mathcal{O},I}^\infty.$$

Combined with Theorem 6.1, this easily implies the claim.  $\square$

Let

$$W_{\text{St}}(G, I) := \sum_{\Pi} \dim_{\overline{\mathbb{Q}}_\ell} (\Pi^\infty)^{K_{\mathcal{D},I}^\infty},$$

where the sum is over the automorphic representations of  $G(\mathbb{A})$  with  $\Pi_\infty = \text{St}_\infty$ . Define  $W_1(G, I)$  similarly. Since  $W_1(G, I)$  is the number of Hecke characters of  $(\mathbb{A}^\infty)^\times$  of conductor dividing  $I$ , it is more or less clear (and will be confirmed by our later calculations) that  $W_1(G, I) + W_{\text{St}}(G, I) \sim W_{\text{St}}(G, I)$ .

It is conjectured that the multiplicities  $m(\Pi)$  of automorphic representations are always equal to 1. If we assume this, then from the preceding discussion we get the following asymptotic estimates:

$$(6.2) \quad h_I \sim h_{I,\eta}^{d-1} \sim d \cdot W_{\text{St}}(G, I).$$

As is explained in [18], cf. pp. 219 and 310 in *loc. cit.*, that  $m(\Pi) = 1$  would follow from the global Jacquet-Langlands correspondence between  $G(F)$  and  $\text{GL}_d(F)$ . This correspondence is proven in the literature for global fields of characteristic zero, cf. [12, Ch.VI]. It is very likely that the theorem is valid in exactly the same formulation in positive characteristic, but the complete proof is still lacking except for  $d = 2$  or  $3$ . To avoid relying on this yet unproven analogue, we will deduce (6.2) from some results about the cohomology of quotients of  $p$ -adic symmetric spaces.

**6.2. Cohomology of varieties having rigid-analytic uniformization.** Let  $\Omega^d$  be Drinfeld's  $(d-1)$ -dimensional symmetric space over  $F_\infty$ . It is obtained from the projective  $(d-1)$ -dimensional space over  $F_\infty$  by removing all rational hyperplanes. In [6], Drinfeld showed that this space has a natural rigid-analytic structure. Let  $\Gamma$  be a discrete, cocompact, torsion-free subgroup of  $\text{PGL}_d(F_\infty)$ .  $\Gamma$  naturally acts on  $\Omega^d$  and the quotient  $X_\Gamma := \Gamma \backslash \Omega^d$  is a proper smooth rigid-analytic variety over  $F_\infty$ , which in fact is the analytification of a projective algebraic variety  $\mathcal{X}_\Gamma$  over  $F_\infty$ ; see [20].

Let  $\mathbb{C}_\infty$  be the completion of an algebraic closure of  $F_\infty$ . Assuming there exists an étale cohomology theory on the category of smooth rigid-analytic spaces satisfying four natural properties [25, pp. 55-56], Schneider and Stuhler computed the groups

$$H^i(X_\Gamma) := H_{\text{et}}^i(X_\Gamma \hat{\otimes}_{F_\infty} \mathbb{C}_\infty, \overline{\mathbb{Q}_\ell}), \quad i \geq 0.$$

Their result implies the following (see Theorem 4 on page 93 of [25]):

**Theorem 6.3.** *Let  $\mu(\Gamma)$  be the multiplicity of the Steinberg representation  $\text{St}_\infty$  in  $L^2(\Gamma \backslash \text{PGL}_d(F_\infty))$ . For  $i \neq d-1$*

$$\dim_{\overline{\mathbb{Q}_\ell}} H^i(X_\Gamma) = \begin{cases} 0, & \text{if } i \text{ is odd or } i > 2(d-1); \\ 1, & \text{if } i \text{ is even}; \end{cases}$$

and

$$\dim_{\overline{\mathbb{Q}_\ell}} H^{d-1}(X_\Gamma) = \begin{cases} d \cdot \mu(\Gamma) + 1, & \text{if } d \text{ is odd}; \\ d \cdot \mu(\Gamma), & \text{if } d \text{ is even}. \end{cases}$$

In [1], Berkovich developed an étale cohomology theory for non-archimedean analytic spaces which satisfies the four properties required in [25]. Moreover, he proved a comparison theorem between analytic and algebraic étale cohomology groups  $H^i(X_\Gamma) \cong H^i(\mathcal{X}_\Gamma \otimes_{F_\infty} \bar{F}_\infty, \overline{\mathbb{Q}_\ell})$ ; see [1, §7.1].

Now we want to apply Theorem 6.3 to  $M_{I,\eta}^\mathcal{P}$ . For this we need to know that the modular varieties  $M_{I,\eta}^\mathcal{P}$  have rigid-analytic uniformization over  $F_\infty$ . As we already mentioned, the reduced norm induces a bijection

$$G(F) \backslash G(\mathbb{A}^\infty) / K_{\mathcal{D},I}^\infty \xrightarrow{\sim} F^\times \backslash (\mathbb{A}^\infty)^\times / \text{Nr}(K_{\mathcal{D},I}^\infty) \cong \mathcal{O}_I^\times / \mathbb{F}_q^\times.$$

Choose a system  $S$  of representatives for this finite coset space. For each  $s \in S$ , let

$$\Gamma_{I,s} := G(F) \cap s K_{\mathcal{D},I}^\infty s^{-1}.$$

**Lemma 6.4.** *Under the natural embedding*

$$\Gamma_{I,s} \hookrightarrow G(F)/F^\times \hookrightarrow G(F_\infty)/F_\infty^\times \cong \mathrm{PGL}_d(F_\infty),$$

$\Gamma_{I,s}$  is a discrete, cocompact, torsion-free subgroup of  $\mathrm{PGL}_d(F_\infty)$ .

*Proof.* It is enough to prove the claim for  $\Gamma_I := \Gamma_{I,1}$ . The fact that  $\Gamma_I$  is a discrete, cocompact subgroup of  $\mathrm{PGL}_d(F_\infty)$  is well-known, cf. [17, Prop. 5.3.8]. We show that it is torsion-free. Assume  $\gamma \in \Gamma_I$  is torsion. Let  $n \in \mathbb{Z}_{>0}$  be its exponent, so  $\gamma^n = 1$ . Since  $\gamma$  is a nonzero element of the division algebra  $D(F)$ , we can consider the finite degree field extension  $K = F[\gamma]$  of  $F$ . We see that  $K$  is obtained by an extension of constants, so  $n$  is coprime to  $p$ . On the other hand,  $\gamma \in K_{\mathcal{D},I}^\infty$ . Choose a place  $x$  which is in the support of  $I$ . Since  $x \notin R$  by assumption, we can consider  $\gamma$  as an element of the principal congruence subgroup  $\Gamma(x) := \{M \in \mathrm{GL}_d(\mathcal{O}_x) \mid M \equiv 1 \pmod{\varpi_x}\}$ . It is not hard to check that  $\Gamma(x)$  has no prime-to- $p$  torsion. Therefore,  $n = 1$ .  $\square$

Next, since

$$(6.3) \quad G(F) \backslash G(\mathbb{A})/K_{\mathcal{D},I}^\infty F_\infty^\times \cong \bigsqcup_{s \in S} (\Gamma_{I,s} \backslash \mathrm{PGL}_d(F_\infty)),$$

we have

$$(6.4) \quad W_{\mathrm{St}}(G, I) = \sum_{s \in S} \mu(\Gamma_{I,s}).$$

**Theorem 6.5.**

$$(M_{I,\eta}^{\mathcal{D}} \otimes_F F_\infty)^{\mathrm{an}} \cong \bigsqcup_{s \in S} (\Gamma_{I,s} \backslash \Omega^d).$$

*Proof.* This follows by applying Raynaud’s “generic fibre” functor to Theorem 4.4.11 in [2], which is stated in the language of formal schemes. The proof of this theorem is only outlined in [2]. Nevertheless, as is shown in [29], the desired uniformization can be deduced from Hausberger’s version of the Cherednik-Drinfeld theorem for  $M_I^{\mathcal{D}}$  [13].  $\square$

Since  $F \hookrightarrow F_\infty$ , by the proper base change theorem we have an isomorphism  $H_{I,\eta}^i \cong H^i(M_{I,\eta}^{\mathcal{D}} \otimes_F \bar{F}_\infty, \bar{\mathbb{Q}}_\ell)$  for all  $i \geq 0$ . Combining this with Theorem 6.3, (6.4), Theorem 6.5 and Berkovich’s results, we easily obtain (6.2). Note also that by comparing the dimensions of cohomology groups in Theorems 6.1 and 6.3, we can deduce that the multiplicities  $m(\Pi)$  are indeed 1.

**6.3. Euler-Poincaré measure.** It remains to estimate  $W_{\mathrm{St}}(G, I)$ . We start with a well-known fact about discrete subgroups of  $p$ -adic groups.

**Theorem 6.6.** *Let  $\Gamma$  be a discrete, cocompact, torsion-free subgroup of  $\mathrm{PGL}_d(F_\infty)$ . Then  $\dim_{\mathbb{Q}} H^0(\Gamma, \mathbb{Q}) = 1$ ,  $\dim_{\mathbb{Q}} H^{d-1}(\Gamma, \mathbb{Q}) = \mu(\Gamma)$ , and  $H^i(\Gamma, \mathbb{Q}) = 0$  for  $i \neq 0, d-1$ . In particular,*

$$(6.5) \quad \chi(\Gamma) := \sum_{i \geq 0} (-1)^i \dim_{\mathbb{Q}} H^i(\Gamma, \mathbb{Q}) = 1 + (-1)^{d-1} \mu(\Gamma).$$

*Proof.* This was first proven by Garland [10] using a discrete analogue of curvature on Bruhat-Tits buildings, but under the assumption that  $q$  is large enough. Casselman [4] removed the restriction on  $q$  by giving a completely different proof which relies on representation-theoretic methods.  $\square$



From (6.5) and (6.4), we get

$$(6.6) \quad W_{\text{St}}(G, I) \sim \sum_{s \in S} (-1)^{d-1} \chi(\Gamma_{I,s}).$$

In [26], Serre developed a theory which allows to compute  $\chi(\Gamma_{I,s})$  as a volume. Serre's result is reproduced in a convenient form in Proposition 5.3.6 of [17]. Combining this with Proposition 5.3.9 in [17], in our situation we get the following statement:

Let  $dg$  be the Haar measure on  $\text{GL}_d(F_\infty)$  normalized by  $\text{Vol}(\text{GL}_d(\mathcal{O}_\infty), dg) = 1$ . Let  $dz$  be the Haar measure on  $F_\infty^\times$  normalized by  $\text{Vol}(\mathcal{O}_\infty^\times, dz) = 1$ . Fix the Haar measure  $dh = dg/dz$  on  $\text{PGL}_d(F_\infty)$  and the counting measure  $d\delta$  of  $\Gamma_{I,s}$ . Then

$$(6.7) \quad \chi(\Gamma_{I,s}) = \text{Vol} \left( \Gamma_{I,s} \setminus \text{PGL}_d(F_\infty), \frac{dh}{d\delta} \right) \cdot \frac{1}{d} \prod_{i=1}^{d-1} \zeta_\infty(-i)^{-1}.$$

The push-forward of the canonical product measure  $d\bar{g}$  on  $G(\mathbb{A})$  to the double coset space  $G(F) \setminus G(\mathbb{A})/K_{\mathcal{D},I}^\infty F_\infty^\times$  induces via (6.3) the measure  $dh/d\delta$  on each  $\Gamma_{I,s} \setminus \text{PGL}_d(F_\infty)$ . Combining this with (6.6) and (6.7), we get

$$(6.8) \quad W_{\text{St}}(G, I) \sim (-1)^{d-1} \text{Vol} \left( G(F) \setminus G(\mathbb{A})/K_{\mathcal{D},I}^\infty F_\infty^\times, d\bar{g} \right) \cdot \frac{1}{d} \prod_{i=1}^{d-1} \zeta_\infty(-i)^{-1}.$$

It remains to compute this volume. It is well-known that  $\|\cdot\| : G(\mathbb{A}) \rightarrow q^\mathbb{Z}$  in (4.1) is surjective. The image of  $F_\infty^\times$  in  $q^\mathbb{Z}$  is clearly  $q^{d\mathbb{Z}}$ . Hence there is an exact sequence

$$0 \rightarrow G(F) \setminus G^1(\mathbb{A})/\mathcal{O}_\infty^\times \rightarrow G(F) \setminus G(\mathbb{A})/F_\infty^\times \rightarrow \mathbb{Z}/d\mathbb{Z} \rightarrow 0,$$

which implies

$$(6.9) \quad \text{Vol} \left( G(F) \setminus G(\mathbb{A})/K_{\mathcal{D},I}^\infty F_\infty^\times, d\bar{g} \right) = d \cdot \#\text{GL}_d(\mathcal{O}_I) \cdot \text{Vol} \left( G(F) \setminus G^1(\mathbb{A}), d\bar{g} \right).$$

Combining (6.2), (6.8), (6.9) and Proposition 4.1, we obtain the following asymptotic formula

$$(6.10) \quad h_I \sim (-1)^{(d-1)} d \cdot \frac{\#\text{GL}_d(\mathcal{O}_I)}{(q-1)} \prod_{i=1}^{d-1} \zeta_F^{R \cup \infty}(-i).$$

Now assume  $I = \text{Spec}(A/\mathfrak{p}) \neq o$  is admissible. Let  $\bar{h}_I$  be the sum of the Betti numbers of  $\bar{M}_{I,o}^{\mathcal{D}}$ . Since the morphism  $M_{I,o}^{\mathcal{D}} \rightarrow \bar{M}_{I,o}^{\mathcal{D}}$  is étale of degree  $[\mathbb{F}_{\mathfrak{p}} : \mathbb{F}_q]$ , (6.10) implies

$$(6.11) \quad \bar{h}_I \sim (-1)^{(d-1)} d \cdot \#\text{PGL}_d(\mathbb{F}_{\mathfrak{p}}) \prod_{i=1}^{d-1} \zeta_F^{R \cup \infty}(-i).$$

On the other hand, by Theorem 5.6 we know that

$$(6.12) \quad \#\bar{M}_{I,o}^{\mathcal{D}}(\mathbb{F}_o^{(d)}) \sim \#\text{PGL}_d(\mathbb{F}_{\mathfrak{p}}) \cdot \prod_{i=1}^{d-1} \zeta_F^{R \cup \infty \cup o}(-i).$$

Combining (6.11) and (6.12), we get

$$\#\bar{M}_{I,o}^{\mathcal{D}}(\mathbb{F}_o^{(d)})/\bar{h}_I \sim \frac{1}{d} (-1)^{d-1} \prod_{i=1}^{d-1} \zeta_o(-i)^{-1} = \frac{1}{d} \prod_{i=1}^{d-1} (q_o^i - 1),$$

which is the second claim of Theorem 1.2. The first claim is Corollary 3.2.

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DEPARTMENT OF MATHEMATICS, PENNSYLVANIA STATE UNIVERSITY, UNIVERSITY PARK, PA 16802

*E-mail address:* `papikian@math.psu.edu`